

Symmetry: a bridge between nature and culture
Simmetria: un ponte tra cultura e natura
Symétries : un pont entre nature et culture

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The possibility of translation implies the existence of an invariant. To translate is precisely to disengage this invariant.[...]

What is objective must be common to many minds and consequently transmissible from one to the other, and this transmission can only come about by [a] “discourse” [...] we are even forced to conclude: no discourse, no objectivity. [...]

Now what is science? I have explained [above], it is before all a classification, a manner of bringing together facts which appearances separate, though they were bound together by some natural and hidden kinship. Science, in other words, is a system of relations. Now we have just said, it is in the relations alone that objectivity must be sought; it would be vain to seek it in beings considered as isolated from one another.

To say that science cannot have objective value since it teaches us only relations, is to reason backwards, since, precisely, it is relations alone which can be regarded as objective.

External objects, for instance, for which the word *object* was invented, are really *objects* and not fleeting and fugitive appearances, because they are not only groups of sensations, but groups cemented by a constant bond. It is this bond, and this bond alone, which is the object in itself, and this bond is a relation.
(Poincaré, 1902, §§4 and 6)

Are symmetries discovered or rather invented by humans ? The stand you may take firmly here reveals a lot of your epistemological position. Conversely, the arguments you may forge for answering to this question, or to one of its numerous narrower or broader variations, shape your whole philosophical thoughts; not specifically about science, by the way. I will try to show how physics helps to (re)consider this issue. Indeed, in the XXth century, physicists have not only extended the notion of symmetry much beyond the rich heritage of geometers but they have also deeply rooted it into the natural world. As a consequence, we can foresee that nature and culture are so coherently entangled one with the other that the two possible answers of what seems an inescapable alternative appear to be two banks continuously connected by one single bridge.

After some brief recalls in § 1 about the articulation between two essential facets of symmetry, namely the notion of transformation and the notion of invariance, I will quickly review in § 2 what kind of transformations physicists talk about. Then, in § 3, I will illustrate, with one of the simplest examples, what is meant by the transformation of a physical law. This will allow us to understand how invariance is indeed a *raison d'être* of the science laws themselves. In the next section, § 4, I will explain how the mathematical work of Emmy Noether and its repercussions in physics has strengthened even more the bind between invariances, that actually make science possible, and the local conservation laws, from which the physical fundamental objects (the quantum particles) come to existence. Eventually, I will conclude in § 5 by some remarks that go beyond physics and concern more generally rational thinking.

1 Transformation and invariance

The modern concept of symmetry has many facets¹ and we shall focus here on two of them only: the notion of *transformation* and the notion of *invariance*.

We shall always require that the set of the transformations we are considering constitutes a *group* G i.e. a set whose any element T is a one to one mapping. Besides, the composition of two elements of G remains an admissible transformation, that is an element of G . Then, any transformation can be undone and the identity (“doing nothing”) is a somehow trivial transformation that belongs to G .

The connexion between transformations and invariance is rather straightforward. If you take an object on which the transformations in G can act, and if

¹Please, see (Mouchet, 2013b) for a more technical paper on the subject where four facets are distinguished and extensively discussed. For a broader audience (in French) see (Mouchet, 2013a).

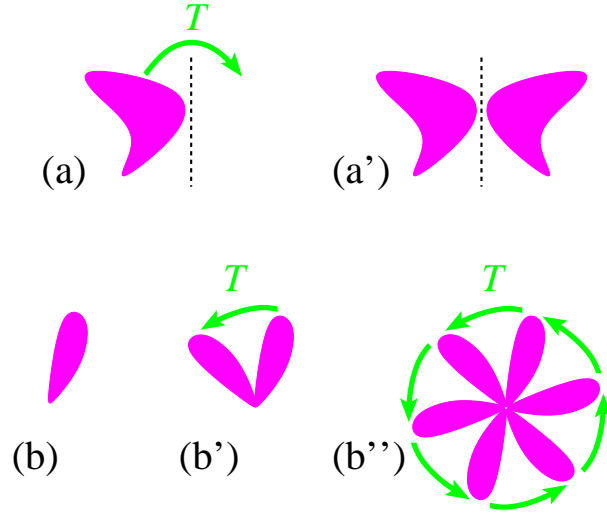


Figure 1: Above, the group of transformations G_a is just the doublet $\{T, T^2 = 1\}$ where 1 denotes the identity and T the mirror symmetry with respect to the dashed line. The pairing of the object (a) and its image by T constitute a *symmetrical* object (a'), that is an object globally invariant by T even if each of its two parts is generally not invariant. Below, the group G_b is made of a rotation of $1/6$ th turn together with its 5 distinct repetitions $\{T, T^2, T^3, T^4, T^5, T^6 = 1\}$. One petal of the flower (b) is not invariant by any rotation in G but the flower (b'') made of 6 petals is.

you collect all the images obtained by applying all the elements of G you will built an object, so called a “symmetrical object”, that remains, by construction, unchanged if you apply on it any transformation in the group. The latter is known as the *symmetry group of the (symmetrical) object*. Of course, one can follow the reverse path: saying that an object is symmetrical under the group G means that it can be reduced in elementary parts differing one from the other by a transformation in G . The figure 1 provides two simple illustrations.

2 Physical transformations

The interplay between transformations and invariance explained in the previous section goes back even before the name of symmetry was forged from the ancient Greek $\sigmaύμ$ (with, concordance, harmony) $μέτρον$ (measure, proportion). This interplay is ubiquitous in nature and art, specially when some parsimony is required (the elaboration of 2 lungs, 6 petals, 12 pillars, 1000 golden mosaic tiles or honey cells, etc.). Even in music and literature, specially for rhythm or rhymed verses, symmetry may help the memory of the bard.

Geometrical transformations like mirror symmetry, rotations or translations are privileged by intuition and actually play a crucial role in our representation of space.

It is seen that experiment plays a considerable role in the genesis of geometry; but it would be a mistake to conclude from that that geometry is, even in part, an experimental science. If it were experimental, it would only be approximative and provisory. And what a rough approximation it would be! Geometry would be only the study of the movements of solid bodies; but, in reality, it is not concerned with natural solids: its object is certain ideal solids, absolutely invariable, which are but a greatly simplified and very remote image of them. The concept of these ideal bodies is entirely mental, and experiment is but the opportunity which enables us to reach the idea. The object of geometry is the study of a particular "group"; but the general concept of group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our sensitiveness, but as a form of our understanding; only, from among all possible groups, we must choose one that will be the standard, so to speak, to which we shall refer natural phenomena.

Experiment guides us in this choice, which it does not impose on us. It tells us not what is truest, but what is the most convenient geometry. It will be noticed that my description of these fantastic worlds has required no language other than that of ordinary geometry. Then, were we transported to those worlds, there would be no need to change that language. Beings educated there would no doubt find it more convenient to create a geometry different from ours, and better adapted to their impressions; but as for us, in the presence of the same impressions, it is certain that we should not find it more convenient to make a change.

(Poincaré, 1895, Conclusions)

In fact, with Galileo Galilei's famous arguments on the invariance of experiments when embarked at constant speed on a boat, since the very beginning of modern physics, transformations are considered that are not pure (static) geometric displacements in the three dimensional Euclidian space. However, in the XXth century, physicists have considered far more general transformations and have worked out much more abstracted geometries in "fantastic worlds":

(i) In the theory of Relativity one builds up a 4-dimensional geometry of space-time that comes with dynamical transformations where time and space coordinates are blended together through linear transformations (the Poincaré group in Special Relativity) or any smooth transformation (the group of space-time diffeomorphisms in General Relativity).

(ii) In quantum theory, transformations act on very abstract spaces made up of algebraic objects — like wavefunctions or operators — that represent quantum

states. In such spaces, a rotation of one turn may have some significant effects (typically on half-integer spins). Transformations can also be made local (the so-called gauge transformations); for instance, a rotation whose angle depends on when and where it is done.

(iii) Transformations may involve the exchange of quantum particles. For instance, one wish to compare the stability of two atomic nuclei, the second being obtained from the first by replacing protons by neutrons and vice-versa. Considering these kinds of permutations is crucial if one wants to understand the existence of antiparticles (very much like in the mirror symmetry in figure 1 that provides a simple rule for constructing the image of an object, the change of the electric charge, grossly speaking, may transform a particle into its antiparticle and conversely) or collective effects (electron shells, laser light, superconductivity, superfluidity, etc.) where quantum particles are indistinguishable in a way that one cannot conceive in our macroscopic world.

3 Transformation of a physical law

The transformations of some physical systems or of their observers (a rotation of a planet, a boost of a boat, a free fall of a lift, the substitution of an electron by a positron) come along with a possible transformation of the physical laws themselves. Take for instance the relation

$$P = 2\pi\sqrt{\frac{\ell}{g}} \quad (1)$$

between the period of oscillations P of a pendulum, its length ℓ and the acceleration of gravity g . In terms of transformations, this formula tells you that if you change the length into $\ell^T = 4\ell$, the period will transform according to $P^T = 2P$. In other words, the relationship between the transformed quantities ($\ell^T = 4\ell, P^T = 2P$) remains the same, namely

$$P^T = 2\pi\sqrt{\frac{\ell^T}{g}}. \quad (2)$$

In fact, the very existence of such a law is a manifestation of some invariance under some transformations. This formula is independent of the date and location of the observation: it remains correct in Venice in 2014 as well as in Pisa in 1583. Even more, it is still exact on the Moon provided we take into account the transformation of g_{Earth} into $g_{\text{Moon}} \simeq g_{\text{Earth}}/6$.

But it is however crucial to remember that a physical law must not be reduced to a formula like (1) alone; it must also come with some domain of validity and, whenever it is possible, a quantitative estimate of the unavoidable uncertainties. We have no need to evoke the “revolutions” of Relativity or quantum physics to rule out a formula like (1); even when remaining within classical

Newtonian mechanics, it becomes inaccurate as soon as we take large variations of the amplitude of the oscillations, or increase the viscosity of the ambient medium. These transformations require a transformation of the equation, since neither amplitudes nor the viscosity appear in (1). It is only when it comes with a domain of validity that a science law may acquire an indelible status; all the more that a more general model or theory helps to delimit its range. Following Zénon, the Renaissance protagonist of Marguerite Yourcenar's novel *The Abyss*,

I have refrained from making an idol of truth, preferring to leave to it its more modest name of exactitude (Yourcenar, 1976, p. 123, A conversation in Innsbruck).

In fact, the whole science consists precisely in filtering from an uncountable set of parameters the very few ones (the relevant variables) that may influence the dynamics of a given system and that allow to make reasonable predictions. If we want, say, to obtain two oscillations in the free air during one second with an amplitude of 5° with a precision up to 10%, formula (1) is sufficient. Even if we take a more precise and therefore more elaborated formula (that takes into account the shape of the pendulum, its amplitude and the ambient viscosity), the mass of the observer, the intensity of the surrounding light, the socio-economic structure that supports the cost of the experiment, the position of the moons of Jupiter still remain irrelevant variables; in other words, the formula is invariant with respect to any realistic transformation of the latter parameters.

The two facets of symmetry recalled in section 1 lie at the source of the *universality* of science laws (including their domain of validity) through two angular stones: the invariance with respect to the transformation of the system (*reproducibility*) and the invariance with respect to the transformation of the observer (*objectivity*).

4 Symmetry and conservation laws

Motivated by the newborn theory of General Relativity, the mathematician Emmy Noether published in 1918 extremely profound results that connect any general (global) invariance to the special (local) invariance with respect to time translation (Kosmann-Schwarzbach, 2010). More precisely the main Noether's theorem stipulates that *for any continuous group of transformations (Lie group) under which an optimization problem is invariant, there exists a conserved quantity*. The continuous groups studied by Sophus Lie are made of transformations labeled by parameters that can vary continuously. For instance, unlike the discrete groups illustrated in figure 1, if we consider every possible angle of rotation (not only the 6 ones of G_6), we recover the continuous group of rotations in the plane whose continuous variable is precisely given by the angle of rotation. By an optimization problem (also known as a variational problem), it is meant that the issue can be formulated by saying that its solutions maximize or minimize

some global quantity². For example, the shape of a soap bubble minimizes its surface, the path followed by a light ray minimizes (or sometimes maximizes) its travel time (Fermat principle of least time). It happens that all the fundamental equations of physics can be derived from such an optimization principle (in classical mechanics, we talk of a principle of least action). By global we mean that the quantity to be optimized depends on a whole set of points (it is an integral) like the total length of a path. Noether's theorem says that invariance of the rules used to compute such global quantity implies the existence of a local conserved quantity, i.e. computed from some quantities attached to one point (and its immediate neighbourhood) of the solution (the orientation of the tangent vector at each point of the shortest path for instance).

As far as I know, all the conservation laws can be seen as consequences of Noether's theorem.

(c1) The invariance under the time shifts implies the existence of a conserved quantity called energy.

In a closed system, energy cannot be neither lost neither created³. There are only conversions (from radiation to electricity, from nuclear energy to thermal energy, from chemical energy to mechanical energy and so on). It was because of an apparent small lack of energy in the β -disintegration, that the existence of the neutrino was inferred by Pauli in 1930. This particle interacts so weakly with the matter that its energy was missed by the detectors.

(c2) The invariance under the space translations implies conservation of the linear momentum (mass \times velocity).

This law is manifest both in the recoil of a canon shooting a bullet and in the recoil of an atom emitting a photon and rules also the collisions on a billiard or in the Large Hadron Collider in CERN.

(c3) The invariance under the space rotations implies conservation of the angular momentum (mass \times velocity \times distance to the rotation center).

Here we take advantage of the isotropy of space: no absolute direction is preferred. These conservation laws govern the way an ice skater controls the rotation speed of her pirouette, imposes helicopters to have a rotor mounted on the tailboom to compensate the changes of angular momentum of the main rotor, makes residue of exploded stars rotate very quickly and form pulsars.

(c4) The invariance under the some specific gauge transformation implies conservation of the electric charge.

This conservation prevents the disintegration of a neutron (non charged) into a proton plus a photon (non charged): here, the charge carried by the proton cannot come out of the blue.

It is worth to notice that such conservation laws are found in the tiniest accessible corners of our world as well as in the largest scales of our universe. Noether's theorem allows to understand this universality: it is directly linked with one of the keystones of science, the reproducibility.

²More precisely, corresponds to a stationnary value of this quantity.

³According to the most famous formula, the mass is just a form of energy and therefore the Lavoisier principle of conservation of mass can be overruled, but this requires a large amount of energy.

Most of the quantum properties are generally extremely fragile and are lost in any measurement process. Quantum particles, even those we consider to be elementary, can be destroyed or created in reactions. In fact, the conserved quantities are the only stable attributes on which one can rely and all the quantum particles appear to be a manifestation of such relatively stable properties.

Elementary particles embody the symmetries; they are their simplest presentations, and yet they are merely their consequence.
(Heisenberg, 1971, chap XX, p. 240)

5 Beyond physics (but without metaphysics): forging concepts and rational thinking

As remarked by Werner Heisenberg, the conservation laws reactivate the “problem of change” (or of becoming) already been settled by pre-Athenian philosophers among which Heraclitus and Parmenides, who proposed the two most extreme solutions along the whole continuous spectrum of possible answers.

For our senses the world consists of an infinite variety of things and events, colors and sounds. But in order to understand it we have to introduce some kind of order, and order means to recognize what is equal, it means some sort of unity. From this springs the belief that there is one fundamental principle, and at the same time the difficulty to derive from it the infinite variety of things.[...] This leads to the antithesis of Being and Becoming and finally to the solution of Heraclitus, that the change itself is the fundamental principle; the ‘imperishable change, that renovates the world,’ as the poets have called it. But the change in itself is not a material cause and therefore is represented in the philosophy of Heraclitus by the fire as the basic element, which is both matter and a moving force.

We may remark at this point that modern physics is in some way extremely near to the doctrines of Heraclitus. If we replace the word ‘fire’ by the word ‘energy’ we can almost repeat his statements word for word from our modern point of view. Energy is in fact the substance from which all elementary particles, all atoms and therefore all things are made, and energy is that which moves. Energy is a substance, since its total amount does not change, and the elementary particles can actually be made from this substance as is seen in many experiments on the creation of elementary particles. Energy can be changed into motion, into heat, into light and into tension. Energy may be called the fundamental cause for all change in the world.

(Heisenberg, 1958, chap. IV)

After all, what Heisenberg wrote for quantum particles (see the end of § 4) can be transposed to any more familiar object or concept. No thoughts, no

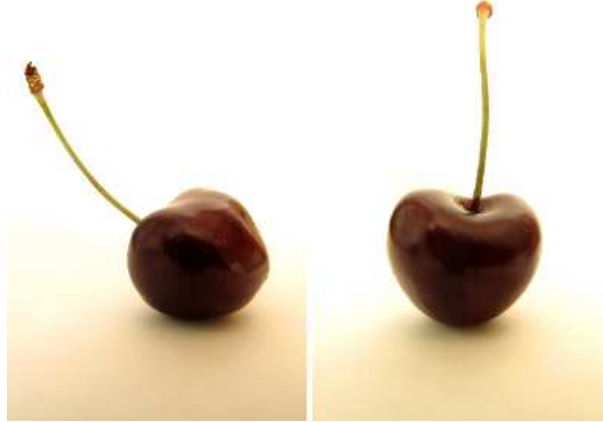


Figure 2: What we call “this cherry” is an equivalence class, i.e. a set of properties like the shape or the color, that remain invariant under an uncountable numbers of transformations of irrelevant parameters like the rotation of its body, some bending of its tail, the orientation of the observer, the intensity of the lighting, the position of the moons of Jupiter, etc. Any concept or fragment of reality requires extraction (abstraction) of some stable properties (at least for a moment long enough to be noticeable). During this process, the physical information has been reduced (simplification) and comes with some validity domain that become fuzzy if too much precision is required (at the nanoscale, the boundary of the cherry is not well defined because, for instance, of the perpetual absorption and desorption of molecules).

language could be possible if we were unable to discern some “constant bond” as Poincaré calls it in the quotation that opens the present work. These stable properties (at least for a moment long enough to be noticeable) reveal what we call the existence of things, building up together what we call the *real world* (including ourselves). A particular macroscopic object like a cherry (figure 2) can be considered as the result of a *simplification* process, an *abstraction* in the etymological sense of the word, where a small cluster of relevant properties have been pruned out from a bunch of irrelevant phenomena. Its shape, taste, colour, size, etc. remain invariant despite the change of light, the variations of the point of view, its rotations, the endless adsorption or desorption of molecules.

With this necessary research of simplicity we come back to the aesthetical primary signification of the word “symmetry”: an elegant efficiency.

Like a pure sound or a melodic system of pure sounds in the midst of noises, so a *crystal*, a *flower*, a *sea shell* stand out from the common disorder of perceptible things. For us they are privi-

leged objects, more intelligible to the view, although more mysterious upon reflection, than all those which we see indiscriminately. They present us with a strange union of ideas: order and fantasy, invention and necessity, law and exception. In their appearance we find a kind of *intention* and *action* that seem to have fashioned them rather as man might have done, but as the same time we find evidence of methods forbidden and inaccessible to us. We can imitate these singular forms; our hands can cut a prism, fashion an imitation flower, turn or model a shell; we are even able to express their characteristics of symmetry in a formula, or represent them quite accurately in a geometric construction. Up to this point we can share with “nature”: we can endow her with designs, a sort of mathematics, a certain taste and imagination that are not infinitely different from ours; but then, after we have endowed her with all the human qualities she needs to make herself understood by human beings, she displays all the inhuman qualities needed to disconcert us... (Valéry, 1964).

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